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The interaction of large amplitude shallow-water waves with an ambient shear flow: non-critical flows

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This paper describes the behaviour of large amplitude, long gravity waves as they move over a horizontal bed into a region where the flow is steady but sheared in a vertical direction. A new class of exact solutions to the nonlinear hydraulic flow equations is derived. These solutions describe progressing waves and are sufficiently general to allow both the shape of the free surface at any instant and the shear profile of the undisturbed flow to be specified. The waves are examples of neutrally stable disturbances in the sense that they neither grow nor decay in amplitude, although, like simple waves on an unsheared flow, they can break.

1. Introduction

The presence of an ambient sheared flow can greatly affect the behaviour of gravity waves in the atmosphere and the oceans. Most of the analyses which describe these effects are, however, valid only when the governing equations can be formally linearized, or when the flow is steady relative to the wave. In this paper we describe the behaviour of a new class of long gravity waves as they propagate over a horizontal bed into a region where the flow is steady but sheared in a vertical direction. Except for the neglect of real-fluid effects, the only approximation used in the analysis is that the waves are so long compared with the fluid depth that the momentum equation in the vertical direction can be replaced by the hydrostatic pressure law. No restriction is placed on the wave amplitudes.

The equations governing shallow-water (hydraulic) shear flows are derived in §2. It can be shown (see Stoker 1957) that, according to these equations, any plane bore-less wave advancing into a region where the fluid is at rest ultimately becomes a simple wave. If x denotes distance in the direction of travel and t denotes time, in any such wave the fluid depth $H(x,t)$ and the horizontal component $u(x,t)$ of fluid velocity satisfy equations of the form

$$\frac{\partial H}{\partial t}+c(H)\frac{\partial H}{\partial x} = 0 \qquad (1.1)$$

and

$$\frac{\partial u}{\partial t}+c(H)\frac{\partial u}{\partial x} = 0. \qquad (1.2)$$

In §4 it is shown that, even when the undisturbed steady flow is sheared in a vertical direction, the hydraulic flow equations still have exact solutions for which

$H(x,t)$ and $u(x,y,t)$ satisfy equations of the form (1.1) and (1.2) even though, now, u depends on the vertical distance y. This is the basic result of the paper. It is established without restricting the form of the undisturbed shear profile.

Of course, unlike the situation when there is no shear, it cannot be argued that H and u must ultimately satisfy equations of the form (1.1) and (1.2) in all waves which advance into a region where the steady flow is sheared in a vertical direction. The waves described are all neutrally stable disturbances: during the passage of the wave the total variations in H, and in u at any fixed height, are identical at all horizontal stations x. The result for H follows from the fact that H satisfies (1.1). For u the result follows from the fact that, according to (1.2),

$$u = \overline{u}(H,y), \tag{1.3}$$

so that u only depends on (x,t) through its dependence on H. However, even though u and H remain bounded, the vertical component of fluid velocity, which has the form

$$v = (\partial H/\partial x)\,\overline{v}(H,y), \tag{1.4}$$

can grow in amplitude as the free surface steepens.

The equations governing $\overline{u}(H,y)$, $\overline{v}(H,y)$ and $c(H)$ are discussed in §4. The boundary conditions for these equations only involve the function which describes the ambient shear profile. The condition at the free surface is universal and is independent of its current shape. This shape can be specified arbitrarily at some fixed time and then calculated from (1.1) at all subsequent times. Since the equations which govern $\overline{u}, \overline{v}$ and c integrate to give the classical Riemann relations when there is no shear, they are called 'the generalized Riemann equations'.

The waves described by our analysis are of two distinct types depending on whether or not the flows they generate contain critical curves along which $u = c$. Those which do are more difficult to analyse. They will be discussed in a subsequent paper. Here, in the first paper, we restrict attention to waves which do not generate critical curves. Then, ultimately, any wave of finite length, which leaves the depth H unchanged after its passage, either overtakes all particles ahead or is overtaken by all particles from behind. The steady shear flows ahead and behind are identical.

In §6 it is shown that the generalized Riemann equations imply that the variations of u with H for any particle satisfies the simple equation

$$[u-c(H)]\,du/dH+g = 0. \tag{1.5}$$

The initial condition for this equation involves the value of u at which the particle enters the wave. This is known in terms of the steady shear flow ahead of or behind the wave. Once the variation of u with H for every particle is known all the other flow variables can be calculated. Unfortunately, however, (1.5) involves the wave speed $c(H)$. This function is not known *a priori* but must be determined at any H by an integral constraint on u over all particles which are currently at the cross-section where the depth is H. The whole problem of describing the interaction of a long wave with an ambient shear flow reduces to solving this mathematical problem.

In §7 it is shown that this problem can easily be solved for small amplitude waves. For, when H differs little from its ambient value H_0, u and c can be represented by a regular power-series expansion in H/H_0-1. The first term in this expansion, with $c(H)$ replaced by $c_0 = c(H_0)$, yields the linear theory established by Burns (1953).

As an example of the interaction of a large amplitude wave with a shear flow, in §5 we consider the special case when the undisturbed flow has constant vorticity. Then, the generalized Riemann equations can be integrated and a full account of the interaction can be given. Details of this interaction are computed when the wave is generated by a discontinuous drop in H at $x = 0$.

2. Hydraulic shear flows

In this paper we describe the behaviour of a class of large amplitude, long gravity waves as they propagate over a horizontal bed into a region where the flow is steady but sheared in the vertical direction. The transmitting fluid is incompressible and has a uniform density. If x denotes horizontal distance in the direction of wave propagation and y denotes vertical distance then, with the conventional interpretation for the symbols, the continuity and momentum equations which govern the flows generated by the waves are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{2.2}$$

and

$$\partial p/\partial y+\rho g = 0. \tag{2.3}$$

The use of the hydrostatic pressure law (2.3), which replaces the momentum equation in the vertical direction, is crucial in establishing the results reported in this paper. However, except for the neglect of real-fluid effects, it is the only approximation used.

The flow direction need not coincide with the direction of wave propagation. Once $u(x,y,t)$ and $v(x,y,t)$ have been determined the horizontal component of fluid velocity transverse to the wave, $u_T(x,y,t)$, is determined from the condition

$$\frac{\partial u_T}{\partial t}+u\frac{\partial u_T}{\partial x}+v\frac{\partial u_T}{\partial y} = 0. \tag{2.4}$$

The flow region is bounded from below by the horizontal bed $y = 0$ and from above by the free surface $y = H(x,t)$. On this free surface the pressure is constant, equal to p_0 say, and consequently (2.3) integrates to give

$$p = p_0+\rho g(H-y). \tag{2.5}$$

When this expression for p is inserted, (2.2) becomes

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial H}{\partial x} = 0. \tag{2.6}$$

Equations (2.1) and (2.6) should be regarded as two equations for the unknowns $u(x,y,t)$ and $v(x,y,t)$. The function $H(x,t)$ must be determined so that the boundary conditions

$$v = 0 \quad \text{on} \quad y = 0 \tag{2.7}$$

and

$$v = \frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} \quad \text{on} \quad y = H(x,t) \tag{2.8}$$

are satisfied.

2.1. *The sigma variables*

In order to simplify the algebra it is convenient to work with the independent variable

$$z = y/H(x,t) \tag{2.9}$$

rather than y, and with the dependent variable

$$w = \frac{Dz}{Dt} = \frac{1}{H}\left\{v - z\left(\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x}\right)\right\} \tag{2.10}$$

rather than v. In terms of these variables (2.1) and (2.6) become, after some algebra,

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + H\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0 \tag{2.11}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + g\frac{\partial H}{\partial x} = 0. \tag{2.12}$$

The boundary conditions (2.7) and (2.8) simplify considerably. Using the definitions (2.9) and (2.10) they become

$$w = 0 \quad \text{on} \quad z = 0,\ 1. \tag{2.13}$$

Variables similar to z and w are often used in meteorology (see Phillips 1957), where they are called 'sigma variables'.

3. Progressing waves on unsheared flow

When the flow is not sheared in the vertical direction

$$\omega = \frac{\partial u}{\partial y} = H^{-1}\frac{\partial u}{\partial z} \equiv 0. \tag{3.1}$$

Then, the only way to satisfy (2.11), (2.12) and the boundary conditions (2.13) is to take

$$w \equiv 0. \tag{3.2}$$

Consequently, $u(x,t)$ and $H(x,t)$ satisfy the classical shallow-water equations (see Stoker 1957)

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + H\frac{\partial u}{\partial x} = 0 \tag{3.3}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial H}{\partial x} = 0. \tag{3.4}$$

Once u and H have been determined from (3.3) and (3.4), it follows from (2.10) and (3.2) that

$$v = z\left(\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x}\right) \tag{3.5}$$

while for any particle

$$z = y/H = \text{constant}. \tag{3.6}$$

The solutions to (3.3) and (3.4) which describe bore-less waves propagating into a region of uniform flow are well known (Stoker 1957). Such waves are always simple waves in which H and $u = U(H)$ satisfy the one-dimensional wave equations

$$\frac{\partial H}{\partial t} + c(H)\frac{\partial H}{\partial x} = 0 \tag{3.7}$$

and

$$\frac{\partial u}{\partial t} + c(H)\frac{\partial u}{\partial x} = 0. \tag{3.8}$$

The wave speed $c(H)$ and $U(H)$ are determined from the Riemann equations

$$H\,dU/dH + (U-c) = 0 \tag{3.9}$$

and

$$(U-c)\,dU/dH + g = 0. \tag{3.10}$$

These follow from (3.3) and (3.4) when the relations (3.7) and (3.8) are used. If

$$u = u_0 \quad \text{when} \quad H = H_0 \tag{3.11}$$

(3.9) and (3.10) integrate, subject to conditions (3.11), to give

$$u = u_0 + 2(gH_0)^{\frac{1}{2}}\,[(H/H_0)^{\frac{1}{2}} - 1] \tag{3.12}$$

and

$$c = u_0 + (gH_0)^{\frac{1}{2}}\,[3(H/H_0)^{\frac{1}{2}} - 2]. \tag{3.13}$$

4. Progressing waves on a sheared flow

The main purpose of this paper is to show that, even when the flow is sheared in the vertical direction, (2.11)–(2.13) also have solutions which describe finite amplitude progressing waves for which $H(x,t)$ and $u(x,z,t)$ still satisfy the one-dimensional wave equations (3.7) and (3.8). For these solutions, however, $c(H)$ is not given by the simple formula (3.13) and w is not identically zero. In fact w has the form

$$w = (\partial H/\partial x)\,W(H,z), \tag{4.1}$$

while u, because it satisfies (3.8), has the form

$$u = U(H,z). \tag{4.2}$$

To determine the equations which govern $U(H,z)$, $W(H,z)$ and $c(H)$ note that, if (3.7), (4.1) and (4.2) are used, (2.11) implies that

$$(-c+U)\frac{\partial H}{\partial x} + H\left(\frac{\partial U}{\partial H} + \frac{\partial W}{\partial z}\right)\frac{\partial H}{\partial x} = 0 \tag{4.3}$$

or

$$H\left(\frac{\partial U}{\partial H} + \frac{\partial W}{\partial z}\right) + (U-c) = 0. \tag{4.4}$$

Similarly, if (3.8), (4.1) and (4.2) are used, (2.12) implies that

$$(U-c)\frac{\partial U}{\partial H}+W\frac{\partial U}{\partial z}+g = 0. \tag{4.5}$$

Equations (4.4) and (4.5) should be regarded as two equations for $U(H,z)$ and $W(H,z)$. The wave speed $c(H)$ must be determined such that the boundary conditions (2.13), which require that

$$W(H,0) = W(H,1) = 0, \tag{4.6}$$

are also satisfied. When there is no shear, $W \equiv 0$ and equations (4.4) and (4.5) reduce to the Riemann equations (3.9) and (3.10). Therefore, in what follows (4.4)–(4.6) are called the generalized Riemann equations (G.R.E.).

4.1. *Supplementary conditions*

Waves of the type described by (3.7), (3.8) and (4.1)–(4.6) can exist adjacent to steady flows which are sheared in the vertical, but not in a horizontal, direction. The simplest situation to imagine is that of a wave, of constant width $\lambda\,(\gg H)$, separating two semi-infinite flow regions in which $H = H_0$, $u = u_0(y/H_0)$ and $u_T = u_{T0}(y/H_0)$. Then, in addition to the boundary conditions (4.6), equations (4.4) and (4.5) must be solved subject to the condition that

$$U = u_0(z) \quad \text{when} \quad H = H_0. \tag{4.7}$$

Conditions (4.6) and (4.7) are not, by themselves, always sufficient to determine a solution to (4.4) and (4.5). To see this note that (4.5) is a first-order quasi-linear equation for $U(H,z)$ with characteristic curves along which

$$\frac{Dz}{DH} = \frac{W}{U-c}, \quad (U-c)\frac{DU}{DH}+g = 0. \tag{4.8}$$

According to the theory of such equations, if H_M and H_m denote the maximum and minimum values of H in the wave, U must be specified at some point on each characteristic curve which lies in the region

$$0 \leqslant z \leqslant 1, \quad H_m \leqslant H \leqslant H_M. \tag{4.9}$$

Consequently, the specification (4.7) is only sufficient if all such characteristic curves also intersect the curve $H = H_0$. The physical meaning of this condition is apparent once it is noted that the characteristic curves are no more than the images of particle paths. For at any particle

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t}+U\frac{\partial H}{\partial x} = (U-c)\frac{\partial H}{\partial x} \tag{4.10}$$

by (3.7), while

$$Dz/Dt = w = W\,\partial H/\partial x \tag{4.11}$$

by (2.10) and (4.1), so that for any particle,

$$Dz/DH = W/(U-c). \tag{4.12}$$

Consequently, condition (4.7) is sufficient if all particles in the wave cross a horizontal station where $H = H_0$ at some time. This occurs if the flow generated by the wave remains wholly subcritical with $U < c$, or wholly supercritical with $U > c$. For subcritical flows all particles in the wave have been in the steady shear flow ahead of the wave at some previous time, for supercritical flows all particles have been in the steady shear flow behind the wave at some previous time. If the flow contains both subcritical and supercritical regions then the wave contains trapped particles which are convected with the wave. These particles need never reach a horizontal station where $H = H_0$ (the corresponding characteristic curves do not intersect the curve $H = H_0$), so that in addition to condition (4.7) some other information on U must also be specified. In this first paper, only flows which do not contain critical curves along which $U = c$ are discussed. Flows with critical curves will be discussed in subsequent papers.

Of course, unlike the situation when $u_0 \equiv$ constant, it cannot be argued that all progressing waves moving into a region where the flow is steady but sheared in the vertical direction are of the type described above. The precise conditions which produce these waves are not understood. However, they can readily be classified by their common properties.

The most important property is that the waves are neutrally stable in the sense that the maximum and minimum values of H and u in the wave at any time are equal to their values at $t = 0$. This property for H follows directly from the fact that $H(x,t)$ satisfies (3.7). For u it follows from the fact that u only depends on (x,t) through its dependence on H. Although $|u|$ and H cannot grow as the wave propagates, $|v|$ and $|w|$ can become infinite. This occurs when the wave profile distorts so much that the factor $\partial H/\partial x$ in (4.1) becomes infinite.

Another important property is that the interaction between the wave and the ambient shear flow is only local. When H returns to H_0 after the passage of the wave, the shear profile returns to the form it had before the arrival of the wave.

5. Formula for $c(H)$ in terms of $U(H, z)$: front speed

If $\partial U/\partial H$ is eliminated from (4.4) and (4.5) the resulting equation can be written as

$$H\frac{\partial}{\partial z}\left(\frac{W}{U-c}\right)+1 = \frac{gH}{(U-c)^2}, \tag{5.1}$$

which integrates, subject to the condition that $W = 0$ when $z = 0$, to give

$$H\frac{W}{U-c}+z = gH\int_0^z \frac{dz}{(U-c)^2}. \tag{5.2}$$

Since $W = 0$ again at $z = 1$, and since we are only dealing with flows for which $U \neq c$, equation (5.2) implies that $c(H)$ must be determined from $U(H,z)$ by the implicit relation

$$gH\int_0^1 \frac{dz}{(U-c)^2} = 1. \tag{5.3}$$

In particular, the speed of the front, $c_0 = c(H_0)$, must be determined from the relation

$$gH_0\int_0^1 \frac{dz}{(u_0(z)-c_0)^2} = 1, \tag{5.4}$$

which is identical to that obtained for small amplitude waves by Burns (1953) and by Freeman & Johnson (1970).

6. Simplification of the G.R.E.

The G.R.E. can be greatly simplified if a slight change of variables is made. As new independent variables we use (H, ψ), rather than (H, z), where the parameter ψ is constant for all particles which outside the wave move horizontally at a height

$$y = H_0 z = H_0 \psi. \tag{6.1}$$

As dependent variables, rather than c, U and W, we use c, U and z.

The equation satisfied by $u = u(H, \psi)$ $[= U(H, z)]$ follows immediately from (4.8), which can be written as

$$(u-c)\,\partial u/\partial H + g = 0. \tag{6.2}$$

According to (4.7) and (6.1) this must be solved subject to the initial condition that

$$u = u_0(\psi) \quad \text{at} \quad H = H_0. \tag{6.3}$$

The equation satisfied by $z(H, \psi)$ is obtained by eliminating W from (4.12) and (5.1) to yield an equation which can be written as

$$\frac{\partial}{\partial H}\left(H \frac{\partial z}{\partial \psi}\right) = \frac{g}{(u-c)^2}\left(H \frac{\partial z}{\partial \psi}\right). \tag{6.4}$$

When this is compared with the equation

$$\frac{\partial}{\partial H}\left(\frac{\partial u}{\partial \psi}\right) = \frac{g}{(u-c)^2}\left(\frac{\partial u}{\partial \psi}\right), \tag{6.5}$$

which is obtained by first dividing (6.2) by $u-c$ and then differentiating with respect to ψ, it follows that

$$\frac{\partial}{\partial H}\left(H \frac{\partial z}{\partial \psi}\bigg/\frac{\partial u}{\partial \psi}\right) = 0. \tag{6.6}$$

This integrates, subject to conditions (6.1) and (6.3), to give

$$H \frac{\partial z}{\partial \psi} = H_0 \frac{\partial u}{\partial \psi}\bigg/ u_0'(\psi), \tag{6.7}$$

which is a special case of the more general condition that in hydraulic flows ω is invariant for a particle. When (6.7) is integrated, subject to the condition that $z = 0$ when $\psi = 0$, it yields

$$y = Hz = H_0 \int_0^{\psi} \frac{\partial u}{\partial \psi} \frac{d\psi}{u_0'(\psi)}. \tag{6.8}$$

In particular, since $z = 1$ when $\psi = 1$ equation (6.8) implies that

$$\int_0^1 \frac{\partial u}{\partial \psi} \frac{d\psi}{u_0'(\psi)} = \frac{H}{H_0}. \tag{6.9}$$

Once $u_0(\psi)$ has been specified, conditions (6.2) and (6.9) furnish two equations for the determination of $c(H)$ and $u(H, \psi)$. Once u has been determined, (6.8) yields $y(H, \psi)$. Since (2.4), which governs the component of fluid velocity transverse to the wave, implies that

$$\partial u_T/\partial H = 0, \quad u_T = u_{T0}(\psi). \tag{6.10}$$

Finally, for completeness, (2.10), (4.1) and (4.8) imply that

$$v = gH_0 \frac{\partial H}{\partial x}(u-c)\int_0^{\psi}(u-c)^{-2}\frac{\partial u}{\partial \psi}\frac{d\psi}{u_0'(\psi)}. \tag{6.11}$$

One of the most significant properties of the large amplitude waves which are described above is that they produce flows in which u, $(\partial H/\partial x)^{-1}v$, ψ and p can be calculated as functions of H and y without knowing how H varies with x and t. This means, for example, that the variation in vertical height of any particle can be calculated as a function of H, the current depth, without knowing how H varies with x and t. It also means that at any fixed x and t the shear profile, that is the variation of u with y, can be determined once the parameter H is known.

7. Small amplitude waves

To obtain some idea of the nature of the flows which are described by the previous analysis we consider the limiting case when H differs little from its ambient value H_0. More precisely, if we define the reference velocity

$$a_0 = \min_{0\leqslant\psi\leqslant 1}|u_0(\psi)-c_0|, \tag{7.1}$$

which is non-zero because the flows we consider contain no critical layers, and if we define the Froude number

$$F = a_0/(gH_0)^{\frac{1}{2}} \tag{7.2}$$

we consider flows for which

$$|H/H_0-1| \ll F^2 \leqslant 1. \tag{7.3}$$

The condition that $F^2 \leqslant 1$ follows from formula (5.4) and the definitions (7.1) and (7.2). These imply that

$$F^2 = \int_0^1 \frac{a_0^2}{(u_0-c_0)^2}d\psi, \tag{7.4}$$

which, because of condition (7.1), is less than or equal to one.

It is convenient to measure all velocities in units of a_0 and to take ψ and

$$h = F^{-2}(H/H_0-1) \tag{7.5}$$

as the basic independent variables. In terms of these variables (6.2) and (6.9) read

$$(u-c)\frac{\partial u}{\partial h}+1 = 0 \tag{7.6}$$

and

$$\int_0^1 \frac{\partial u}{\partial \psi}\frac{d\psi}{u_0'(\psi)} = 1+F^2h. \tag{7.7}$$

These equations are to be solved subject to the conditions that

$$c = c_0, \quad u = u_0(\psi) \quad \text{when} \quad h = 0. \tag{7.8}$$

When condition (7.3) holds, so that

$$|h| \ll 1, \tag{7.9}$$

the solutions to (7.6) and (7.7) which satisfy conditions (7.8) can be expanded in power series in h. When the expansions

$$u = u_0(\psi) + \sum_{n=1}^{\infty} h^n u_n(\psi) \tag{7.10}$$

and

$$c = c_0 + \sum_{n=1}^{\infty} c_n h^n \tag{7.11}$$

are inserted in (7.10) and (7.11) it can readily be shown that $u_n(\psi)$ is a polynomial of degree $2n-1$ in the variable

$$s(\psi) = [u_0(\psi) - c_0]^{-1}. \tag{7.12}$$

Note that, because in the measurement scale adopted $a_0 = 1$,

$$|s(\psi)| \leqslant 1. \tag{7.13}$$

The coefficients in the polynomial expressions for the $u_n(\psi)$ are given in terms of the constants c_m, which are themselves determined from condition (7.7) in terms of integral powers of $s(\psi)$ over the full range of ψ. The algorithm for determining these constants is a marching process. First u_1 is determined, then c_1; then the third-degree polynomial u_2, then c_2; and so on. In particular,

$$u_1 = -s, \quad c_1 = -\frac{3}{2}\int_0^1 s^4\, d\psi \bigg/ \int_0^1 s^3\, d\psi \tag{7.14}$$

while

$$u_2 = -\tfrac{1}{2}s^3 - \tfrac{1}{2}c_1 s^2 \tag{7.15}$$

and

$$c_2 = -\left[\frac{15}{4}\int_0^1 s^6\, d\psi + 5c_1 \int_0^1 s^5\, d\psi + \tfrac{3}{2}c_1^2 \int_0^1 s^4\, d\psi\right] \bigg/ \int_0^1 s^3\, d\psi. \tag{7.16}$$

When the expansion (7.10), with u_1 and u_2 given by equations (7.14) and (7.15), is inserted, (6.8) yields the expression

$$\bar{y} = \frac{y}{H_0} = \psi + h\int_0^{\psi} s^2\, d\psi + h^2 \int_0^{\psi} (\tfrac{3}{2}s^4 + c_1 s^3)\, d\psi + \ldots \tag{7.17}$$

for $y(h, \psi)$. If (7.17) is used to determine ψ as a function of y and h, the expansion for u can also be written as

$$u = u_0(\bar{y}) - h\left[u_0'(\bar{y})\int_0^{\bar{y}} s^3(\psi)\, d\psi + s(\bar{y})\right] + O(h^2). \tag{7.18}$$

The coefficient of h in (7.18) is identical with that obtained by Freeman & Johnson (1970). They formally linearized the governing equations and looked for solutions of the form

$$u - u_0(\bar{y}) = h(x - c_0 t)\, \eta(\bar{y}). \tag{7.19}$$

To the same order of approximation (6.11) and (7.17) imply that the vertical component of fluid velocity

$$v = (\partial h/\partial x)\, \beta(\bar{y}), \tag{7.20}$$

where

$$\beta = (u_0(\bar{y}) - c_0)\int_0^{\bar{y}} \frac{d\psi}{(u_0 - c_0)^2} \tag{7.21}$$

satisfies the low frequency (zero wavenumber) Rayleigh equation

$$(u_0 - c_0)\,\beta'' - u_0''\beta = 0. \tag{7.22}$$

8. Interaction of a large amplitude progressing wave with a linear shear flow

As a simple example of the interaction of a large amplitude wave with a shear flow we consider the special case when ahead of the wave

$$u = \omega y = \omega H_0 \psi \quad (\equiv u_0(\psi)). \tag{8.1}$$

Then, (5.4) implies that

$$c_0 = \tfrac{1}{2}\omega H_0[1 + \operatorname{sgn}\omega(1+4g/\omega^2 H_0)^{\frac{1}{2}}] \quad (\geqslant 0). \tag{8.2}$$

Since $c_0 - u_0 > 0$ for both positive and negative ω, the flow ahead of the wave is wholly subcritical.

When $u_0 = \omega H_0 \psi$, equations (6.2) and (6.9) simplify considerably. Equation (6.9) immediately integrates to give the relation

$$u_A(H) - u_B(H) = \omega H \tag{8.3}$$

between

$$u_A = u(H, 1), \quad u_B = u(H, 0). \tag{8.4}$$

Also, since u_A and u_B denote the variations of u for the two distinct particles $\psi = 1$ and $\psi = 0$ they both satisfy (6.2). Accordingly,

$$(u_A - c)\, du_A/dH + g = 0 \tag{8.5}$$

and

$$(u_B - c)\, du_B/dH + g = 0. \tag{8.6}$$

Equations (8.3), (8.5) and (8.6) are three equations governing the variations of u_A, u_B and c as functions of H. They must be solved subject to the initial conditions that

$$c = c_0, \quad u_B = 0, \quad u_A = \omega H_0 \quad \text{when} \quad H = H_0. \tag{8.7}$$

Once $u_B(H)$ has been determined, (6.8) implies that

$$u = \omega y + u_B(H). \tag{8.8}$$

When the expression (8.1) for u_0 and the expression (8.8) for u are used, (6.11) implies that

$$v = \frac{gy}{u_B - c}\frac{\partial H}{\partial x}. \tag{8.9}$$

Equations (8.3), (8.5) and (8.6) can readily be solved. In terms of the variable θ, which is defined by the condition that

$$\sinh\theta = \tfrac{1}{2}\omega(H/g)^{\frac{1}{2}}, \tag{8.10}$$

the solutions can be written as

$$(\omega/g)\,[u_A - c(0)] = 2\theta + e^{2\theta} - 1, \quad (\omega/g)\,[u_B - c(0)] = 2\theta + 1 - e^{-2\theta}, \tag{8.11}$$

and

$$(\omega/g)\,[c - c(0)] = 2(\theta + \sinh 2\theta). \tag{8.12}$$

In terms of θ_0, which is defined by the condition that

$$\sinh\theta_0 = \tfrac{1}{2}\omega(H_0/g)^{\frac{1}{2}}, \tag{8.13}$$

$$(\omega/g)\, c_0 = e^{2\theta_0} - 1, \quad (\omega/g)\, c(0) = e^{-2\theta_0} - 1 - 2\theta_0. \tag{8.14}$$

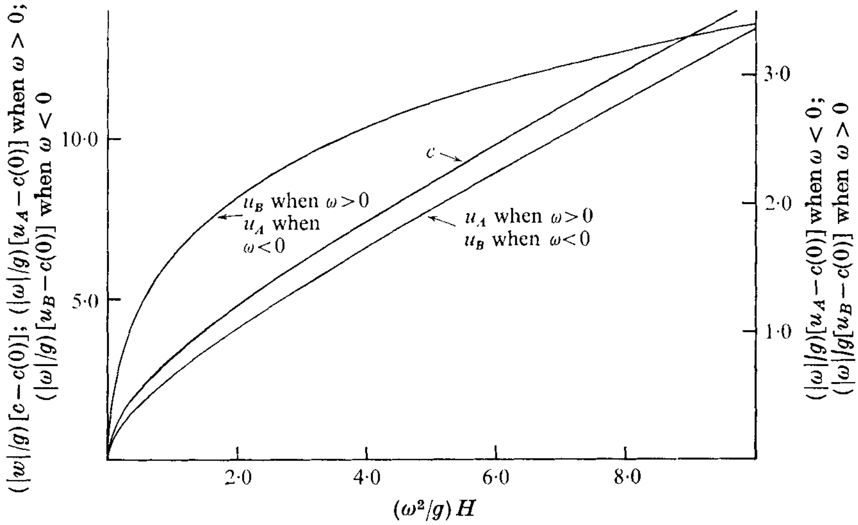

FIGURE 1. Variations in $(\omega/g)[u_A-c(0)]$, $(\omega/g)[u_B-c(0)]$ and $(\omega/g)[c-c(0)]$ as functions of $(\omega^2/g)H$.

In equations (8.10)–(8.14),

$$\operatorname{sgn}\theta = \operatorname{sgn}\omega. \tag{8.15}$$

The variations of $(\omega/g)[u_A-c(0), u_B-c(0), c-c(0)]$ with $(\omega^2/g)H$, which are described by (8.10)–(8.12), are depicted in figure 1. Note that since

$$\left.\begin{aligned}\frac{dc}{dH} = \omega\left(\frac{1+2\cosh 2\theta}{2\sinh 2\theta}\right), \quad \frac{du_A}{dH} = \omega e^{2\theta}(e^{2\theta}-1)^{-1},\\ \text{and} \qquad du_B/dH = \omega(e^{2\theta}-1)^{-1},\end{aligned}\right\} \tag{8.16}$$

c, u_A and u_B increase as H increases.

The classical Riemann relations (3.12) and (3.13) can be obtained from the relations (8.10)–(8.14) in the limit as $\omega \to 0$ but H remains finite. In this limit $\theta, \theta_0 \to 0$ while, according to (8.10), (8.13) and (8.14),

$$\theta/\omega \to \tfrac{1}{2}(H/g)^{\frac{1}{2}}, \quad \theta_0/\omega \to \tfrac{1}{2}(H_0/g)^{\frac{1}{2}}, \quad c(0) \to -2(gH_0)^{\frac{1}{2}}. \tag{8.17}$$

When these results are used, (8.11) and (8.12) predict that

$$\left.\begin{aligned}(u_A, u_B) &\to 2(gH_0)^{\frac{1}{2}}[(H/H_0)^{\frac{1}{2}}-1]\\ c &\to (gH_0)^{\frac{1}{2}}[3(H/H_0)^{\frac{1}{2}}-2]\end{aligned}\right\} \quad \text{as} \quad \omega \to 0. \tag{8.18}$$

To calculate the variation in height y of a particle as a function of H, note that for a particle

$$\frac{Dy}{Dt} = v = \frac{gy}{u_B-c}\frac{\partial H}{\partial x} \tag{8.19}$$

by (8, 9) and

$$\frac{DH}{Dt} = (u-c)\frac{\partial H}{\partial x} = (\omega y + u_B - c)\frac{\partial H}{\partial x} \tag{8.20}$$

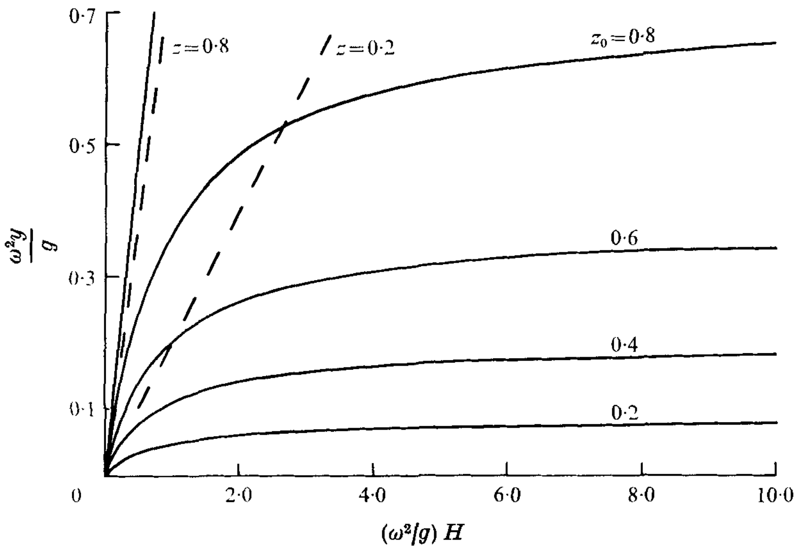

FIGURE 2. Variations in $(\omega^2/g)\,y$ for different particles as functions of $(\omega^2/g)\,H$. The broken rays denote the variations of y with H for the limiting case of no shear.

by (8.8), so that
$$\frac{Dy}{DH} = \frac{gy}{(u_B - c)\,(\omega y + u_B - c)}. \tag{8.21}$$

In terms of $z = y/H$ and θ condition (8.22) reads
$$\frac{Dz}{D\theta} = \frac{4z(z-1)}{1+(1-2z)\tanh\theta}. \tag{8.22}$$

When $\omega = 0$, for a particle
$$z = y/H = \text{constant} = z_0,\ \text{say}. \tag{8.23}$$

Figure 2 depicts the variations of $(\omega^2/g)\,y$ with $(\omega^2/g)\,H$ for several particles. As $(\omega^2/g)\,H \to 0$ $(\theta \to 0)$ these trajectories asymptote to those given by (8.23), which are depicted by broken rays in figure 2. Note that as $(\omega^2/g)\,H \to \infty$ all particles move horizontally.

8.1. *Particle paths in a centred expansion wave*

As an illustration of the actual particle motions which can occur in the flows described in this section, we calculate the trajectories of particles on the free surface during the passage of a centred expansion wave.

In an expansion wave which is centred at $x = 0$ at $t = 0$, $H(x,t)$, which satisfies (3.7), is determined from the condition
$$x = c(H)\,t, \quad \text{so that} \quad \partial H/\partial x = [t\,c'(H)]^{-1}. \tag{8.24}$$

When the expression (8.24) for $\partial H/\partial x$ is used, (8.20) implies that at the free surface, where
$$y = H, \quad t\frac{DH}{Dt} = \frac{\omega H + u_B - c}{c'(H)}. \tag{8.25}$$

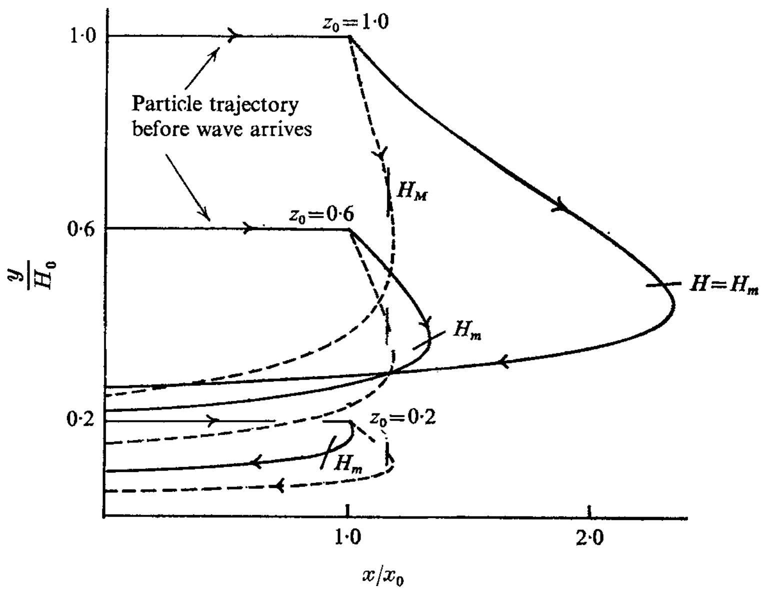

FIGURE 3. Particle trajectories in a linear shear flow before and after they are overtaken by a wave centred at $x = 0$. – – –, particle trajectories in an unsheared flow which has the same undisturbed depth and mean velocity.

Once the variation of H with t for a particle on the free surface has been calculated from the last of equations (8.25), the variation of y with t follows from the first of these equations, and the variation of x with t from (8.24). In terms of $\sigma = e^{2\theta}$, equation (8.25) integrates to give

$$\frac{t}{t_0} = \frac{f(\sigma)}{f(\sigma_0)}, \quad \text{where} \quad f(\sigma) = \frac{\sigma}{(\sigma-1)^3}e^{-\sigma} \tag{8.26}$$

and t_0 denotes the time at which the particle is traversed by the front of the wave for which $\sigma = \sigma_0$. In terms of σ, according to (8.10)

$$(\omega^2/g)\,y = (\sigma-1)^2/\sigma, \tag{8.27}$$

while, according to (8.24), (8.12) and (8.14),

$$\frac{\omega x}{g t_0} = \left[\frac{1-\sigma_0}{\sigma_0} + \ln\frac{\sigma}{\sigma_0} + \frac{\sigma^2-1}{\sigma}\right]\frac{f(\sigma)}{f(\sigma_0)}. \tag{8.28}$$

In terms of the parameter σ_0, the Froude number of the undisturbed shear flow (based on the mean speed) is

$$F = \tfrac{1}{2}|\omega|H_0/(gH_0)^{\frac{1}{2}} = \tfrac{1}{2}|\sigma_0 - 1|\sigma_0^{-\frac{1}{2}}. \tag{8.29}$$

In (8.26)–(8.29),

$$1 < \sigma < \sigma_0 \quad \text{when} \quad \omega > 0; \qquad 0 < \sigma_0 \leqslant \sigma < 1 \quad \text{when} \quad \omega < 0. \tag{8.30}$$

The trajectory of a particle on the free surface is shown in figure 3 for the special case when $F = \frac{1}{2}$. For comparison, the trajectory of a particle on the free surface of an unsheared flow which has the same ambient Froude number F is also depicted. The equation of this trajectory can be written as

$$\frac{x}{x_0} = (1+F)^{-1}\left[F + 3\left(\frac{y}{H_0}\right)^{\frac{1}{2}} - 2\right]\left(\frac{y}{H_0}\right)^{-\frac{3}{2}}, \tag{8.31}$$

where x_0 denotes the station at which the particle is overtaken by the front of the centred wave. Equation (8.31) can be obtained from (8.27) and (8.28) as

$$(\omega, \sigma, \sigma_0) \to 0.$$

Figure 3 also depicts the trajectories of particles which are not on the free surface when they are overtaken by the wave. Each particle is identified by the value of $y/H = z_0$, say, when it was overtaken by the wave. The portions of these trajectories which are actually traversed depend on the amplitude of the wave. Typical portions which correspond to a centred wave taking the flow from the depth H_0 to a depth $H_m < H_0$ are marked. Thereafter, if the flow is maintained at a constant depth H_m, the particles move horizontally.

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